Mathematics and Redistricting

Trinity College, Fall 2021 (Kyle Evans)

Mathematics and Fairness

In what ways is mathematics used to define “fairness”?

- Elementary school –
- Middle school –
- High school –

Mathematics and Fairness

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Mathematics and Fairness

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- Elementary school – Equal
- Middle school – Proportional
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Mathematics and Fairness

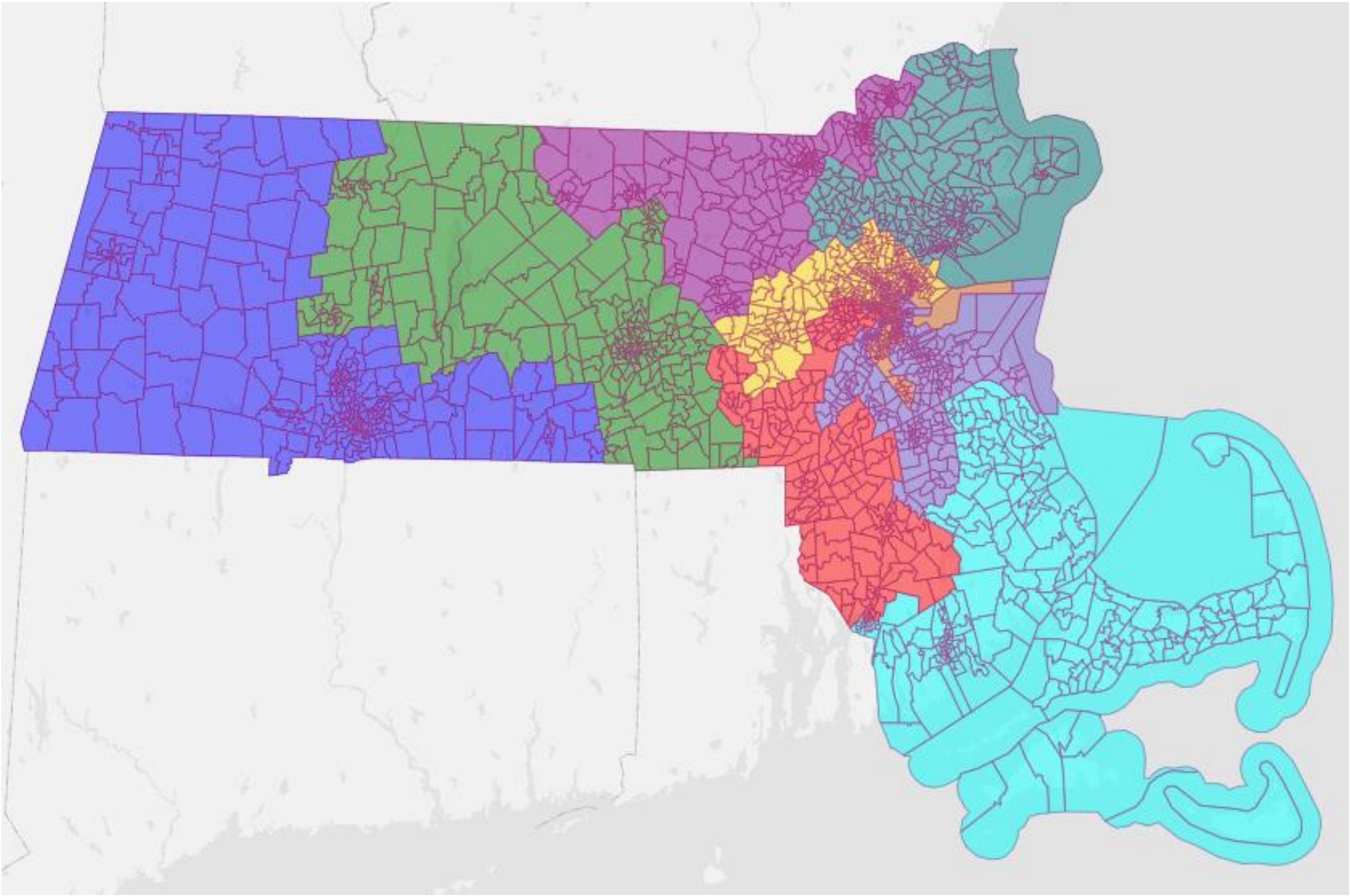
In what ways is mathematics used to define "fairness"?

- Elementary school – Equal
- Middle school – Proportional
- High school – True to a claim or standard
  "Unfair" = outlier

Mathematics and Fairness

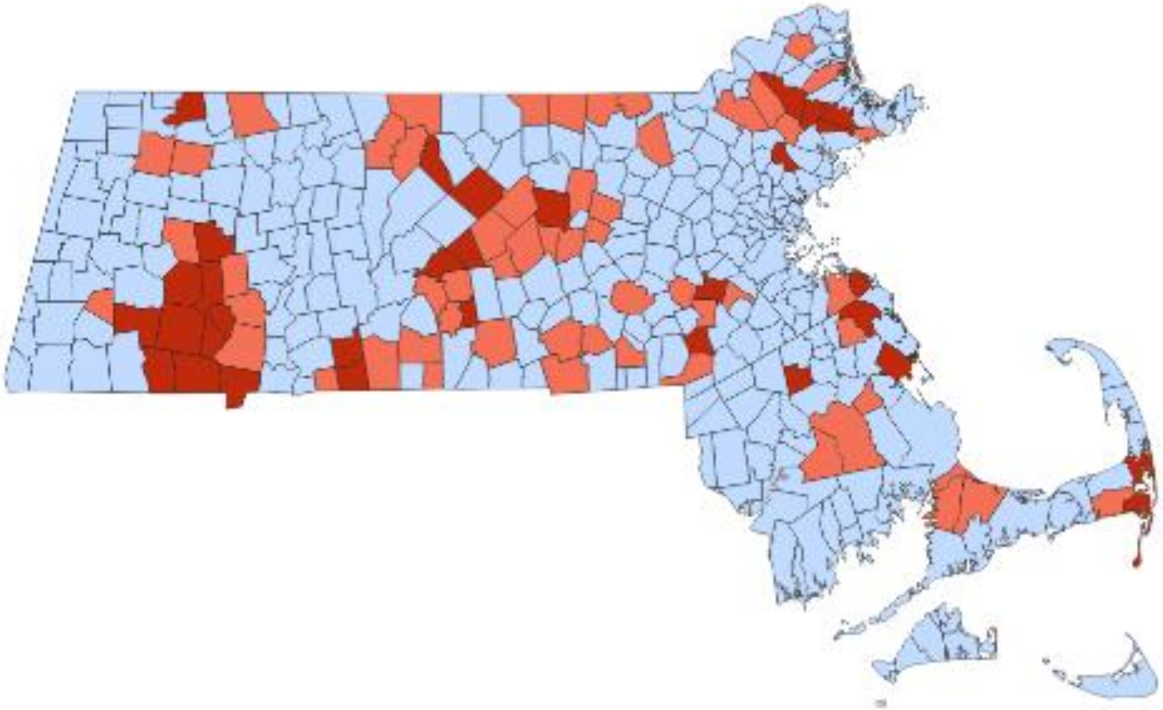
What's the appropriate question to ask about the "fairness" of a district plan?

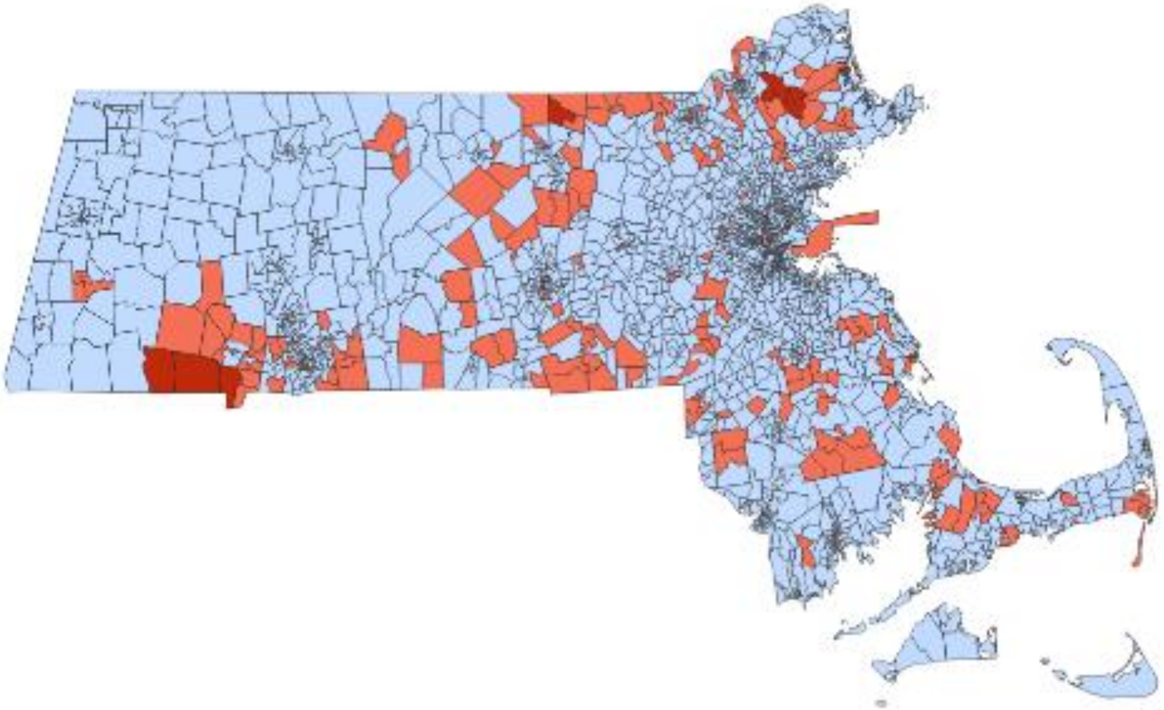
- Proportionality – should representation be somewhat proportional to a state's voters?
  (Measures of partisan bias such as the efficiency gap, mean-median difference, seats/votes curve all relate to some notion of proportionality)



Massachusetts (Congressional)

- Republicans earn between 30%-40% of votes for statewide elections
- Massachusetts has 9 congressional districts (10 in previous decade)
- Currently Dems have a 9-0 advantage in representatives
- In fact, the last time Reps won a single district was in 1994
- Dems have won 125 consecutive House elections in Massachusetts
- Maps have never been challenged for partisan gerrymandering





Massachusetts (Congressional)

- Republicans earn between 30%-40% of votes for statewide elections
- Dems have won 125 consecutive House elections in Massachusetts
- Maps have never been challenged for partisan gerrymandering
- Why? It's simply based on where Republicans live in the state!
- "There are more possible district plans than particles in the galaxy" and every single one of them would produce a 9-0 Democrat map!

Mathematics and Fairness

What's the appropriate question to ask about the "fairness" of a district plan?

- Proportionality – should representation be somewhat proportional to a state's voters?
  (Measures of partisan bias such as the efficiency gap, mean-median difference, seats/votes curve all relate to some notion of proportionality)

- ★ Outlier analysis – is a district map an outlier relative to other possible district maps? ★

Mathematicians and Redistricting

The following three groups of mathematicians have had the strongest impact on math research and court cases in redistricting:

- Metric Geometry and Gerrymandering Group (Tufts/MIT)
- Quantifying Gerrymandering (Duke)
- Princeton Gerrymandering Project

Understanding the basics of some of their work can give us valuable insight into how advanced mathematical approaches are (now commonly) applied to redistricting maps and court cases.

Probability and Expected Value

If you roll a standard 6-sided die, what is the probability of rolling each side?

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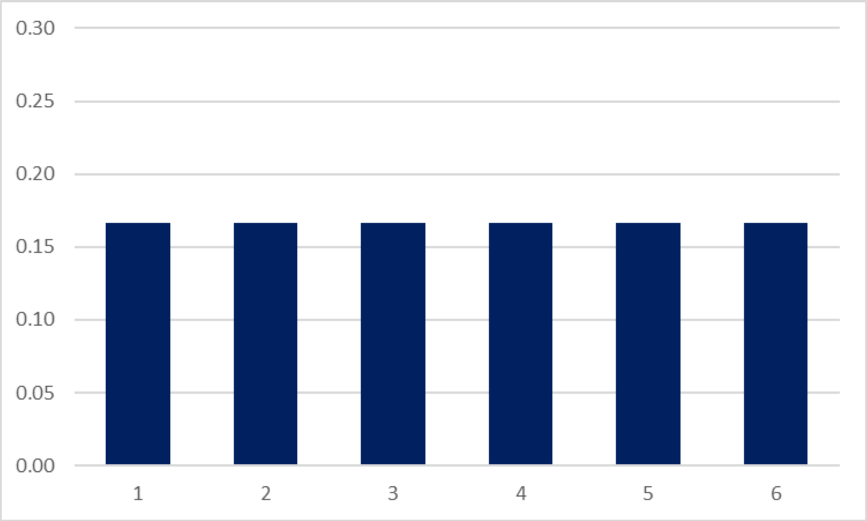


This is an example of a theoretical probability distribution (what is expected to happen – the actual likelihood).

Probability and Expected Value

Suppose you roll a 6-sided die 25 times and record the results.

1: 5 times

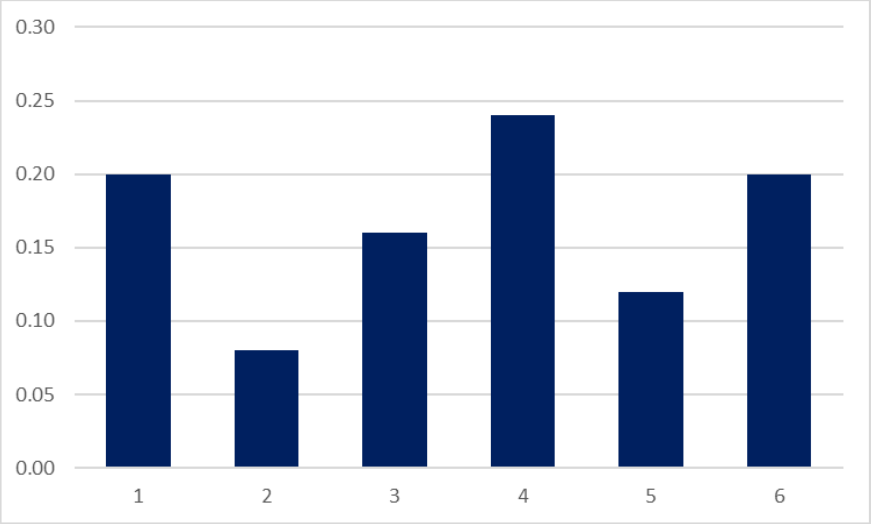
2: 2 times

3: 4 times

4: 6 times

5: 3 times

6: 5 times



This is an example of an empirical probability distribution (based on your sample/experiment).

Probability and Expected Value

The expected value is the weighted average of all possible outcomes.

In rolling a 6-sided die, each number 1-6 occurs 1/6 of the time so

$$\mathrm{EV} = \left(1 \cdot \frac{1}{6}\right) + \left(2 \cdot \frac{1}{6}\right) + \left(3 \cdot \frac{1}{6}\right) + \left(4 \cdot \frac{1}{6}\right) + \left(5 \cdot \frac{1}{6}\right) + \left(6 \cdot \frac{1}{6}\right) = 3.5$$

Probability and Expected Value

Suppose our 25 rolls of the die from before happened in order as

6, 5, 1, 3, 6, 3, …

How does the average change with each roll?

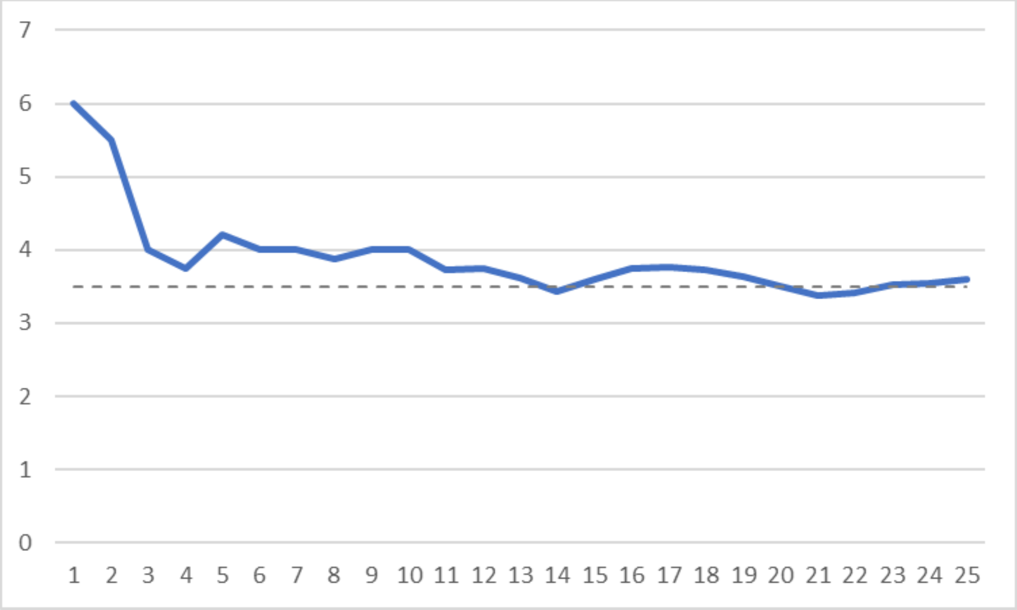
First roll: 6 Average = 6

Second roll: 5 Average = (6+5)/2 = 5.5

Third roll: 1 Average = 12/3 = 4

Fourth roll: 3 Average = 15/4 = 3.75

Fifth roll: 6 Average = 21/5 = 4.2



Law of Large Numbers

As the number of trials increases, empirical probability distributions will approach the theoretical probability distribution and any calculated values will get closer and closer to the expected or average values.

Probability Simulations

(1) Roll two 6-sided dice and record their sum.
Perform 20 trials and count how many of each result you obtained.

(2) Draw 3 cards from the same suit (13 cards) from a standard deck.
You “win” if the cards are in ascending order and “lose” otherwise.
Perform 10 trials and record how many wins you obtained.

(3) Generate and plot 20 random (x, y) points where x and y are between 0 and 1. Count how many are inside and outside the drawn arc.

Monte Carlo Methods

In general, there are many things that are quite difficult to calculate exactly, but in which (many) samples can easily be taken and calculated to approximate the true value.

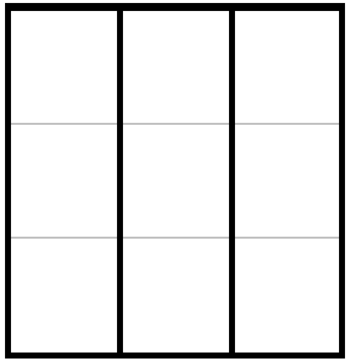
The general processes of repeated random sampling to estimate a value are called Monte Carlo methods.

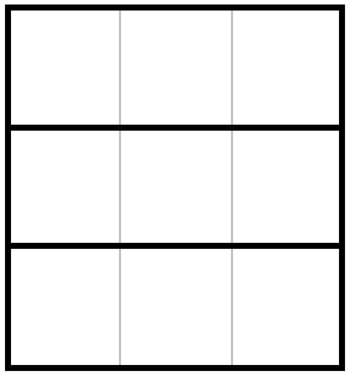
These methods are applied in many fields, including computer science, statistics, mathematics, biology, engineering, finance, law, etc.

The big idea is to generate a distribution and see if a particular map is an outlier within the distribution. Let's start in a scenario where we can determine the entire distribution.

Grid examples

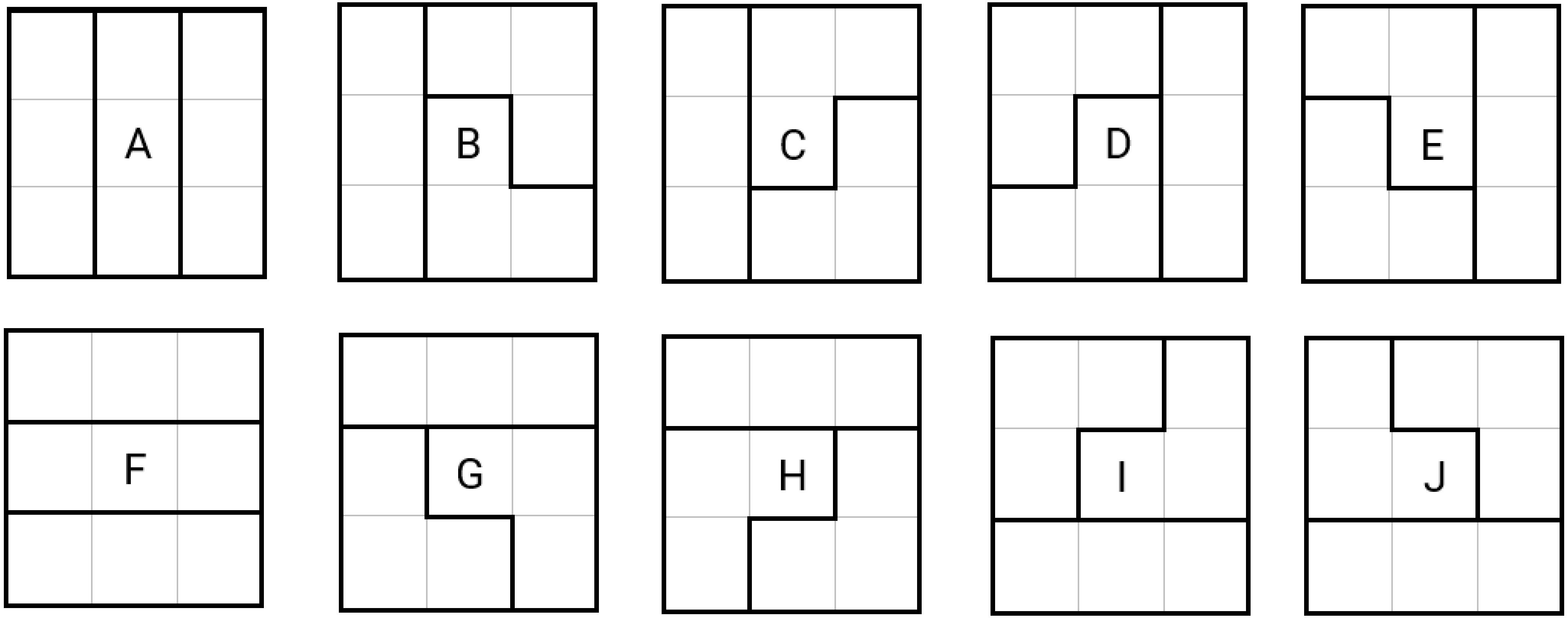
Let's start simple: Consider all 3x3 grids (maps) divided into 3 equal and connected pieces (districts).





Can you draw all 10 possible grids?

Grid examples



Partisan Considerations

What about the partisan nature of the map?

Suppose the grid has the following distribution of Stars and Diamonds:

♦	♦	★
♦	♦	★
★	★	★

Then for every possible districting plan, we can see how well each party performs.

Scale of the Problem

In our 3x3 example, we found every possible grid and we could do something similar in the 5x5 case using technology.

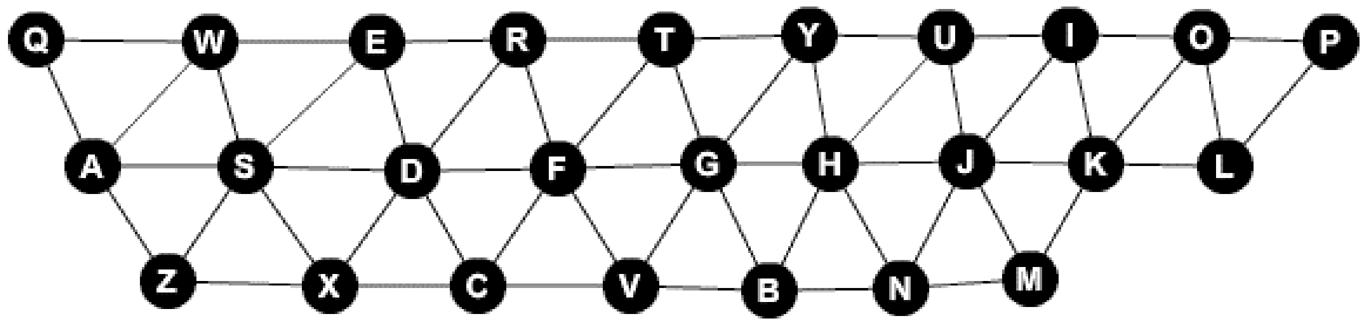
https://mggg.org/metagraph/5x5.html

It turns out there are 4,006 ways to divide a 5x5 grid into 5 connected shapes of 5 blocks each. We could distribute the parties and ask two different questions:

1) How does this arrangement of voters perform over all 4,006 plans? Is ours an outlier?
2) How does this plan perform for all possible arrangements of that many voters of each party? Is ours an outlier?

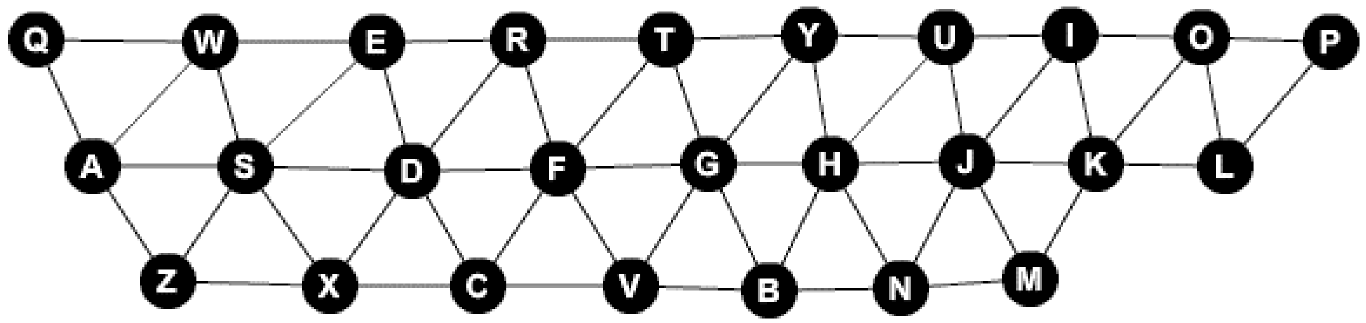
Random Walks

Suppose an ant was to walk along the keys of a keyboard. The graph below shows a keyboard and which keys are next to each other.



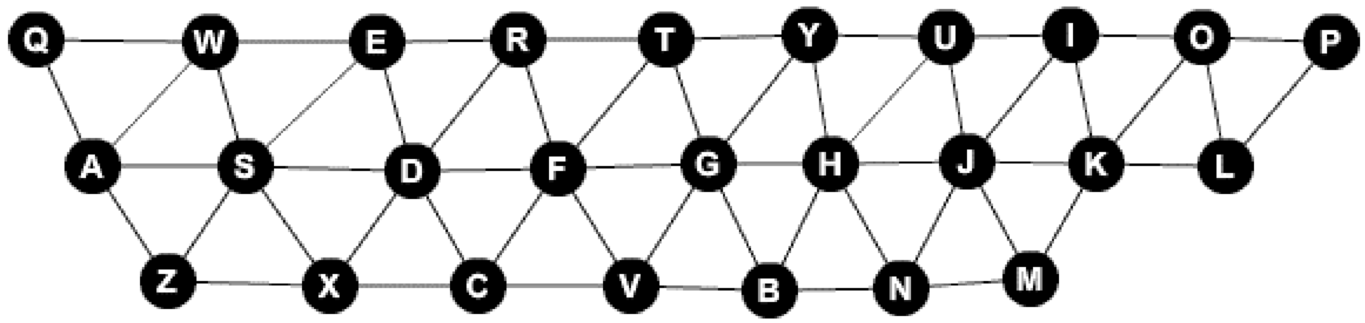
Random Walks

Let's say the ant started on the letter Q. They could then travel to A or W, with a 50% chance of each. Let's say heads is A, tails is W.



Random Walks

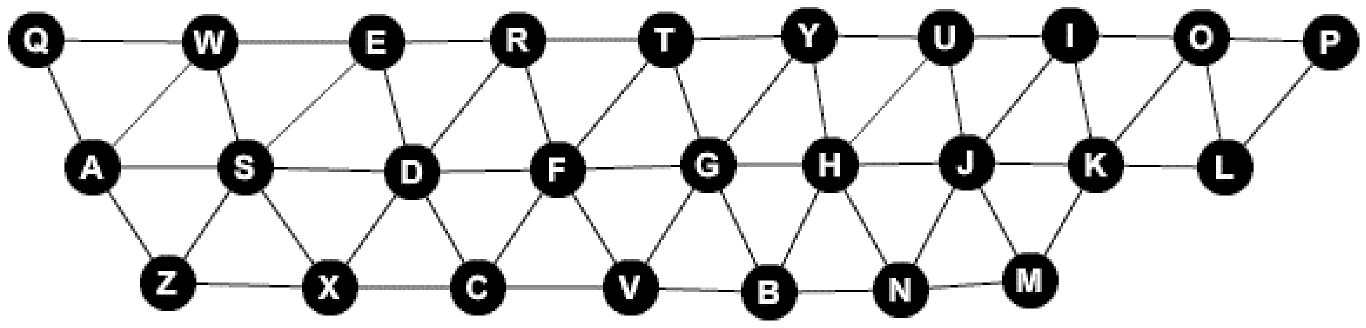
You flip a tails and proceed to W. There is now a 25% chance of going to each of the four letters Q, A, S, E (it doesn't matter that the ant was just on Q, all directions are equally likely).



Random Walks

There are multiple types of questions we might be interested in:

- What is the distribution of letters after a lot of steps?
- What is the distribution of letters that are reached after n steps?



Simulate two different random walks on this keyboard graph.

Markov Chains

Once again, there are many types of random distributions that are difficult to calculate exactly but can be approximated by a sequence of random actions.

The general processes of simulating sequences with random actions that only depend on your current location (previous actions have no effect on current) are called Markov Chains.

These methods are applied in scenarios involving the prediction of behavior, such as Google's PageRank algorithm or iPhone's auto-fill text.

Markov Chains

Important facts:

- No matter what point you start the random walk at, if you take a lot of steps and perform many trials, the distribution of points will approach the true distribution.
- And the points with a higher degree (the number of lines connected to it) will show up more often in the random walk than points with a lower degree.

MCMC and Redistricting

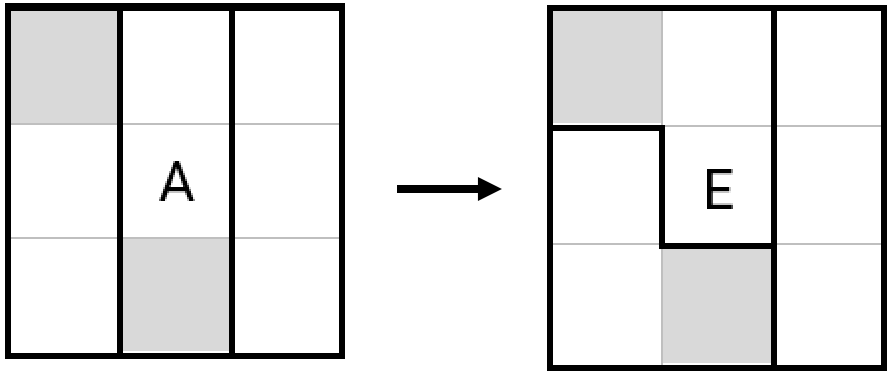
Mathematical methods that use ideas from Monte Carlo methods and from Markov Chains at the same time are called MCMC methods.

These techniques are *necessary* for redistricting problems because of the impossibly large size of valid districting plans.

So how are simulations and random walks on graphs related to district maps and judging whether a district map is fair or unfair?

Flip move

How do we “move” from one district plan to another? Randomly swap the districts of two boxes if a new valid districting plan is formed.



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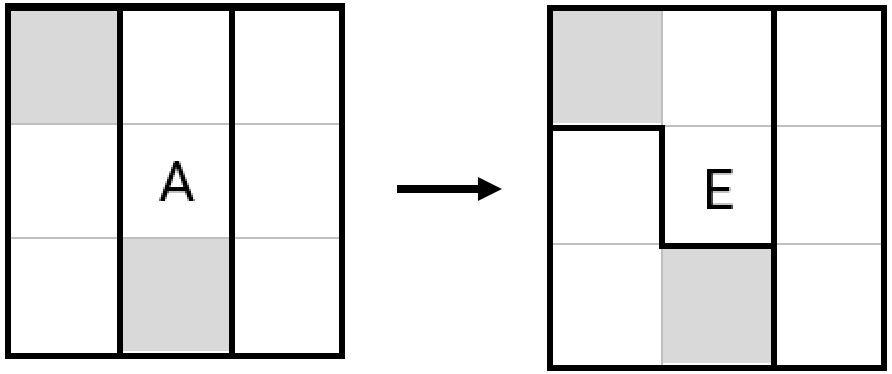

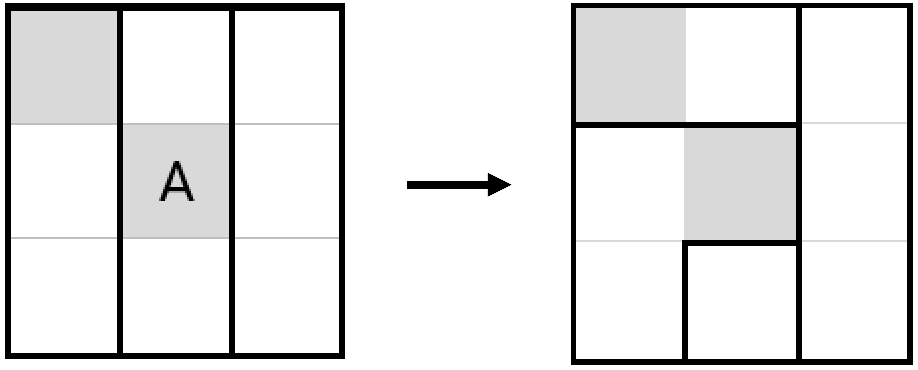

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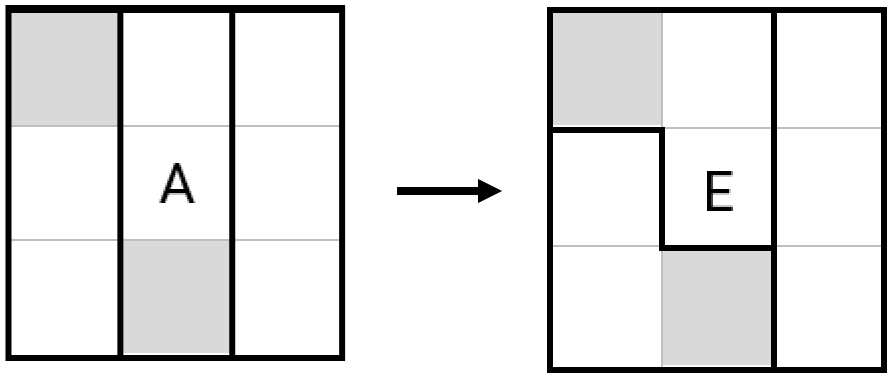
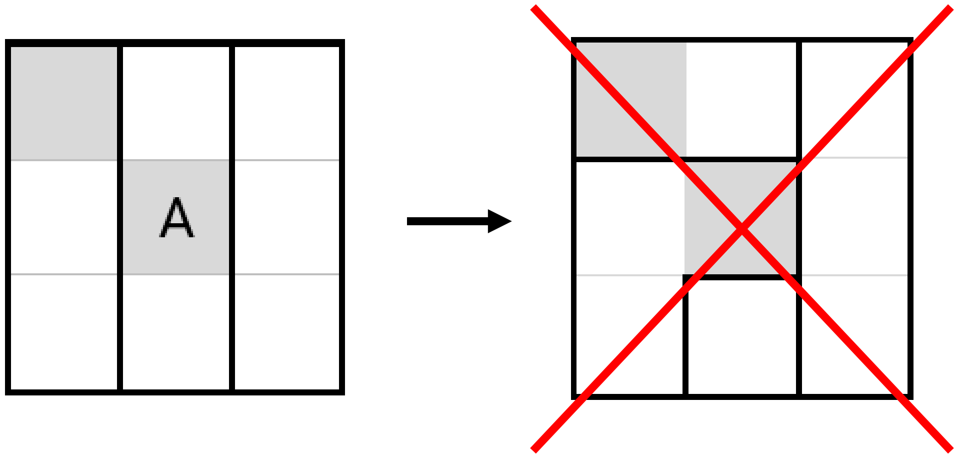
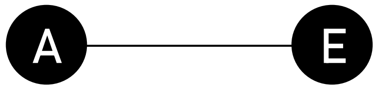




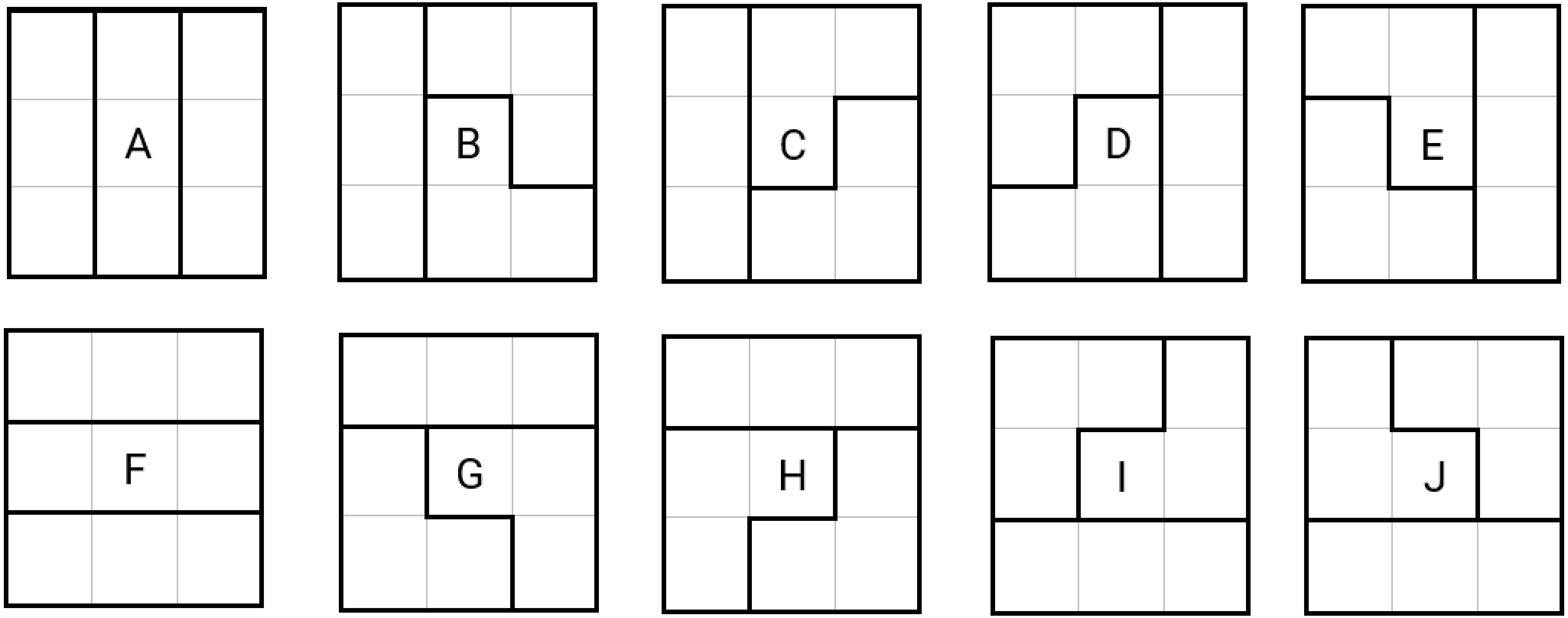
Metagraph

We can then create a metagraph (graph of graphs) with graphs connected if they can be reached from a single swap move.



Can you find other pairs of graphs from our list of 10 that are separated by a single swap move?

Grid examples



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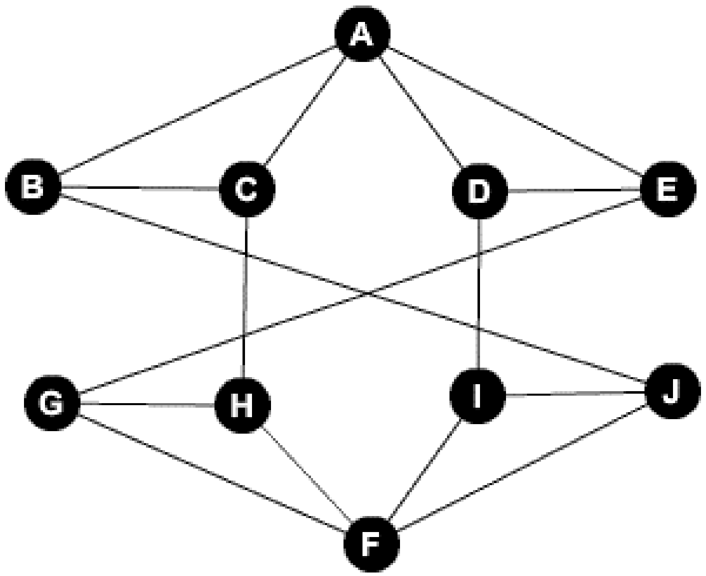
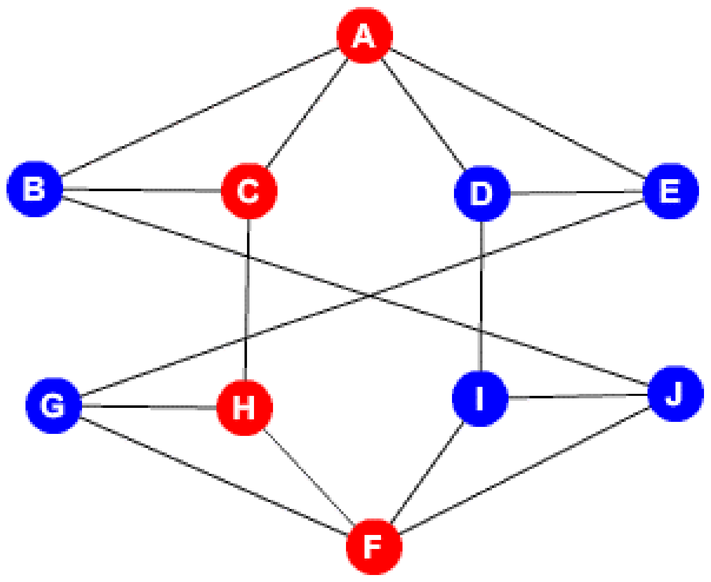


Partisan Considerations

Looking at the metagraph, the grids in which ★ wins more districts are colored in blue and the grids in which ♦ wins more districts are colored in red.

You could then take a random walk on this graph and see how well each party performs over a large number of steps (trials).



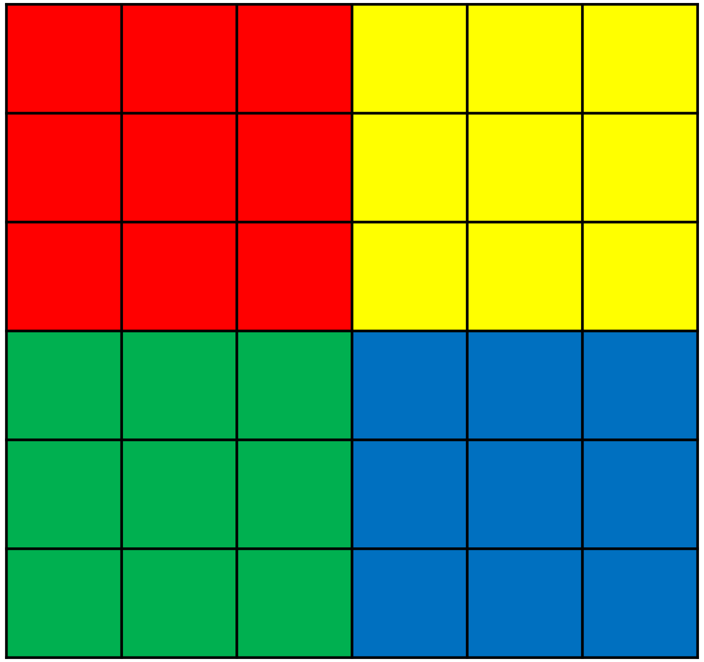
Flip Method

- Start with the actual map (look at party performance).
- Randomly flip the district assignment of 2 blocks in a way that maintains contiguity of districts.
- Repeat this process thousands of times and build a distribution of party performance and see how the actual map compares.

- (Random walks) With enough trials, the distribution approaches the actual distribution no matter where you start!

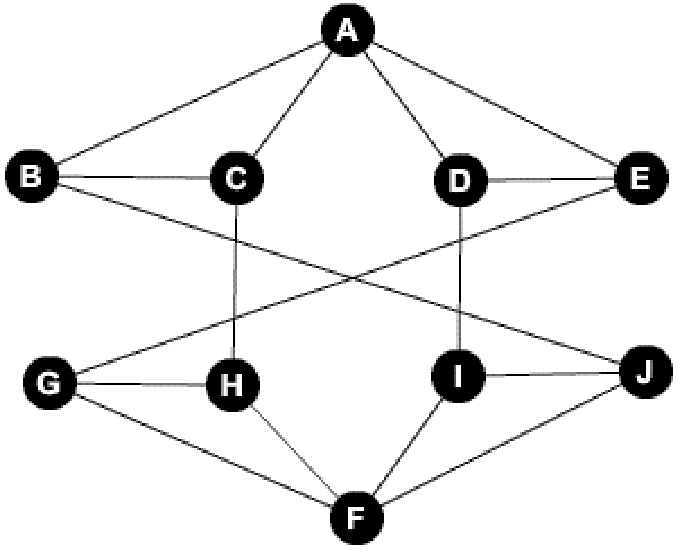
Your Turn

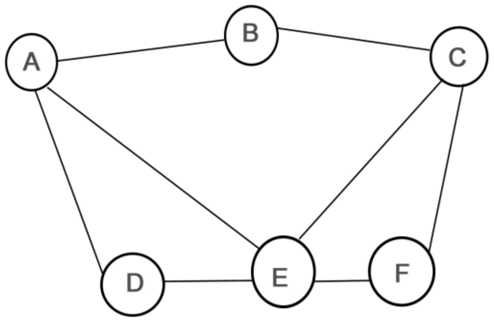
- Apply 4 steps of the Flip algorithm to the district grid below.

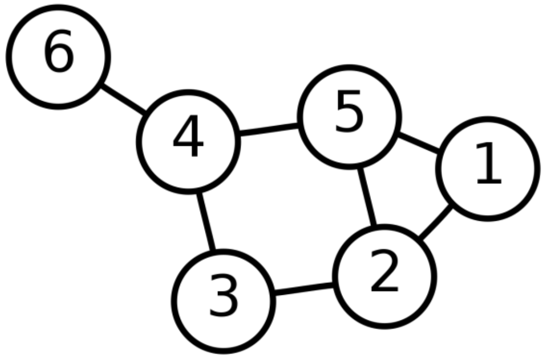


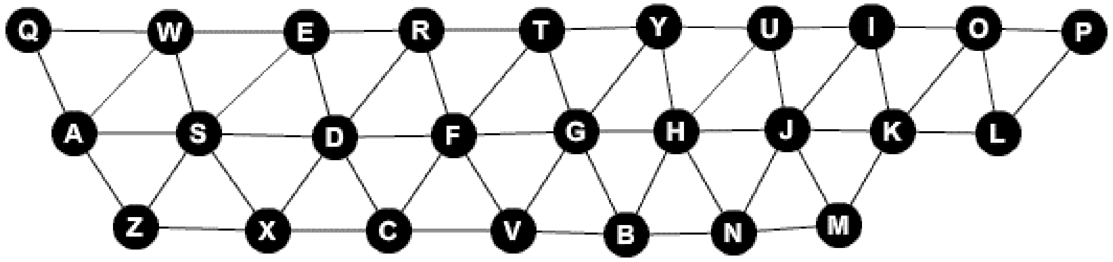
Redistricting as Graphs

- Graph Theory – study of graphs, made up of vertices and edges









Redistricting as Graphs

- A map can be drawn as a graph
- Units of geography (such as counties, voting precincts, or Census blocks) become vertices
- Edges are drawn between vertices if they border each other on the map (can directly drive from one place to another)


Litchfield
Hartford
Tolland
Windham
New Haven
Middlesex
New London
Fairfield
Long Island Sound


Litchfield
Hartford
Tolland
Windham
New London
Middlesex
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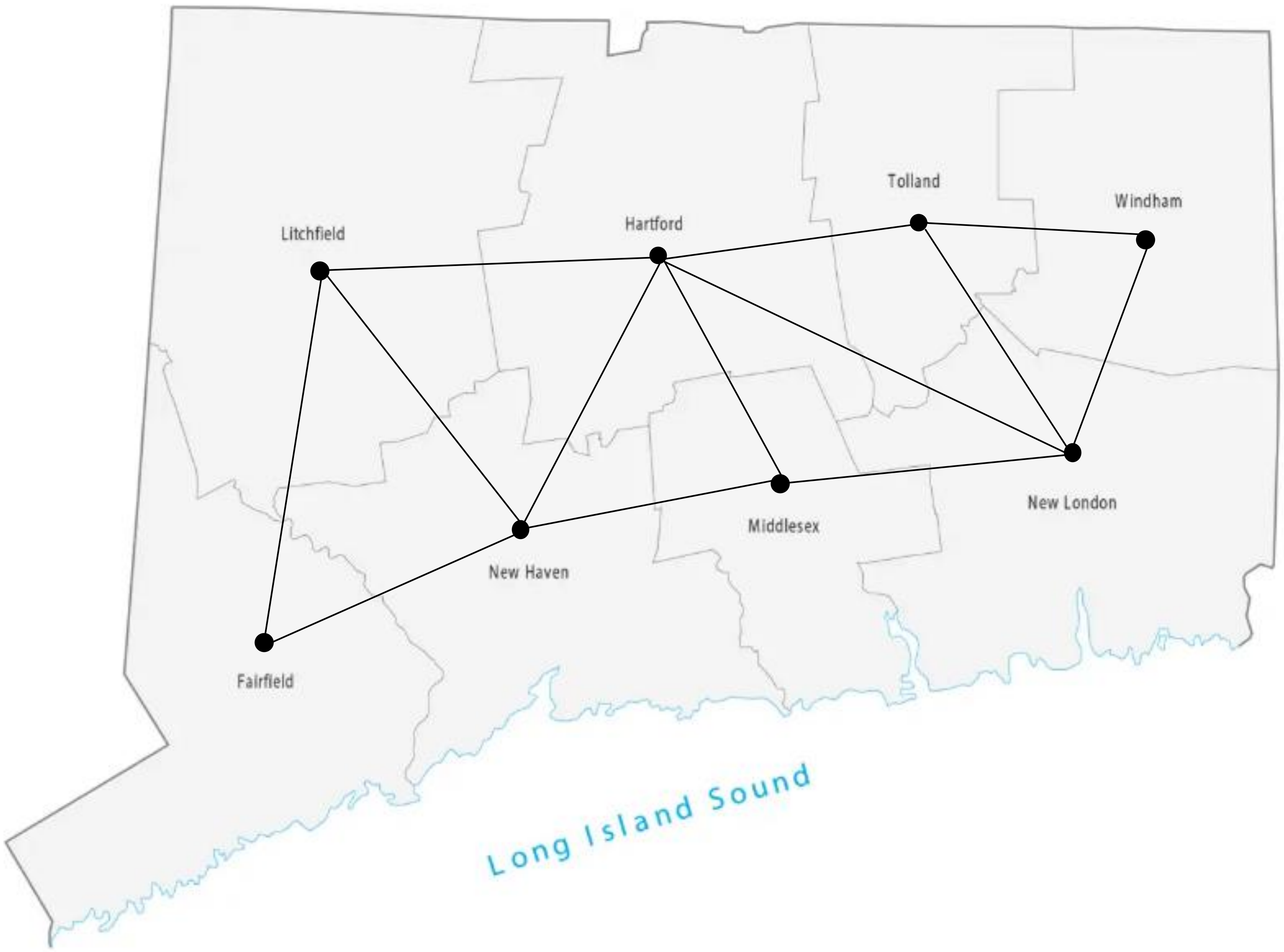
Litchfield
Hartford
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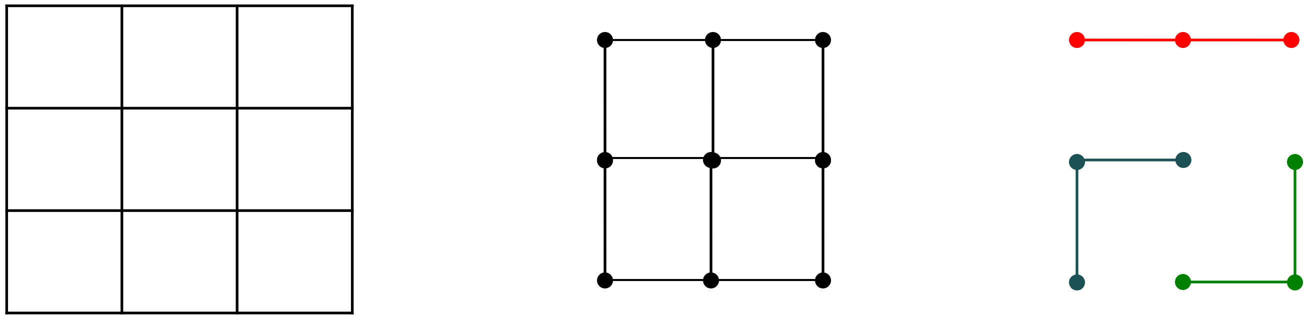
Long Island Sound

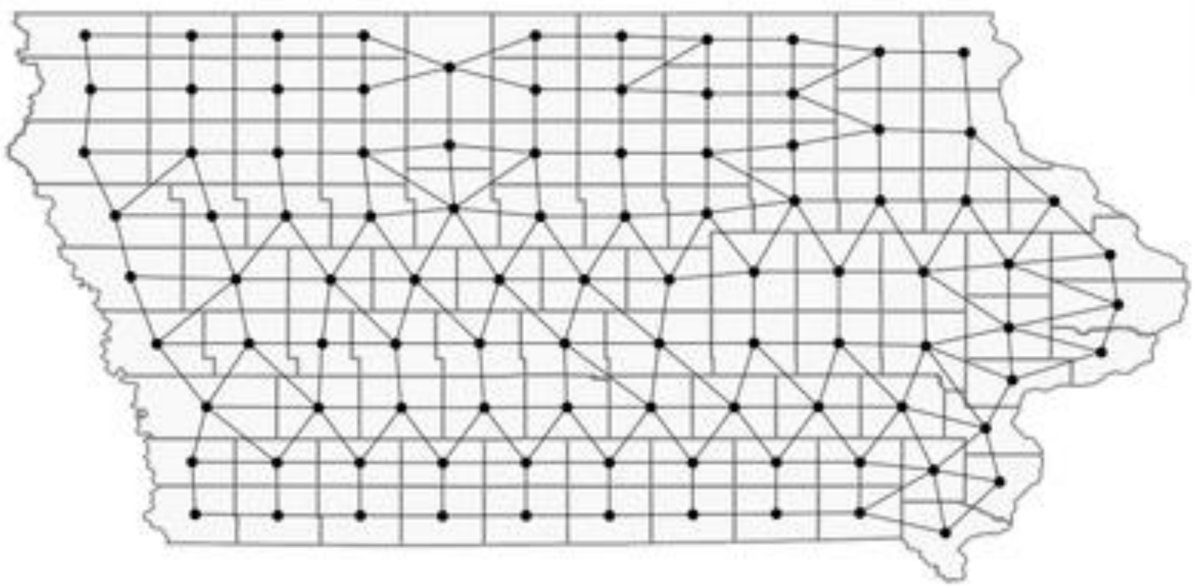
Redistricting as Graphs

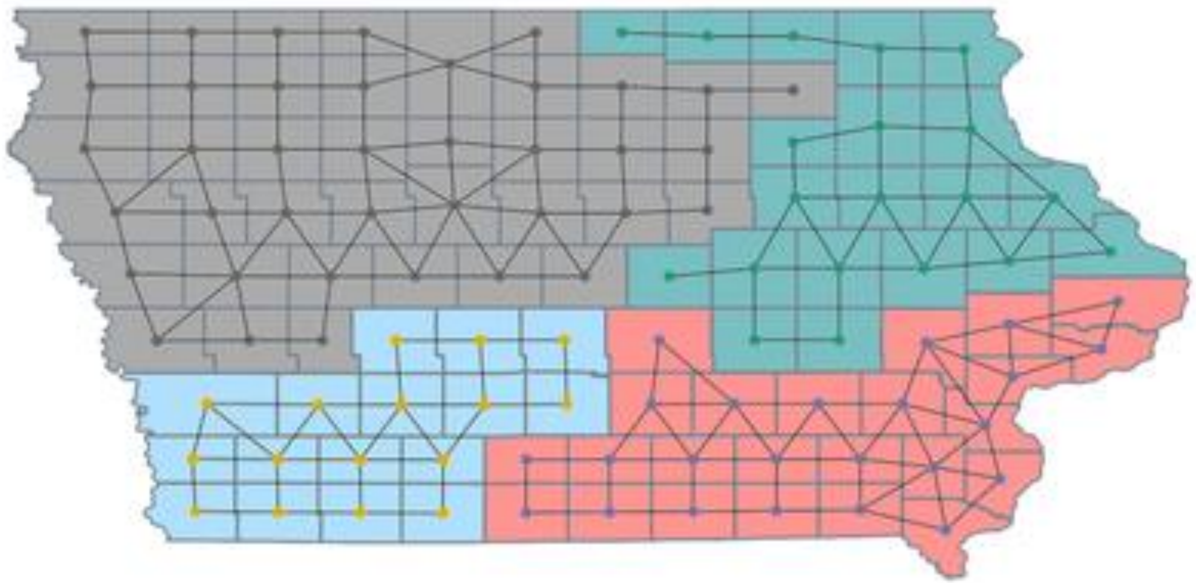
- The "redistricting problem" can be thought of as a graph theory problem – split a connected graph (state) into n connected subgraphs (districts)



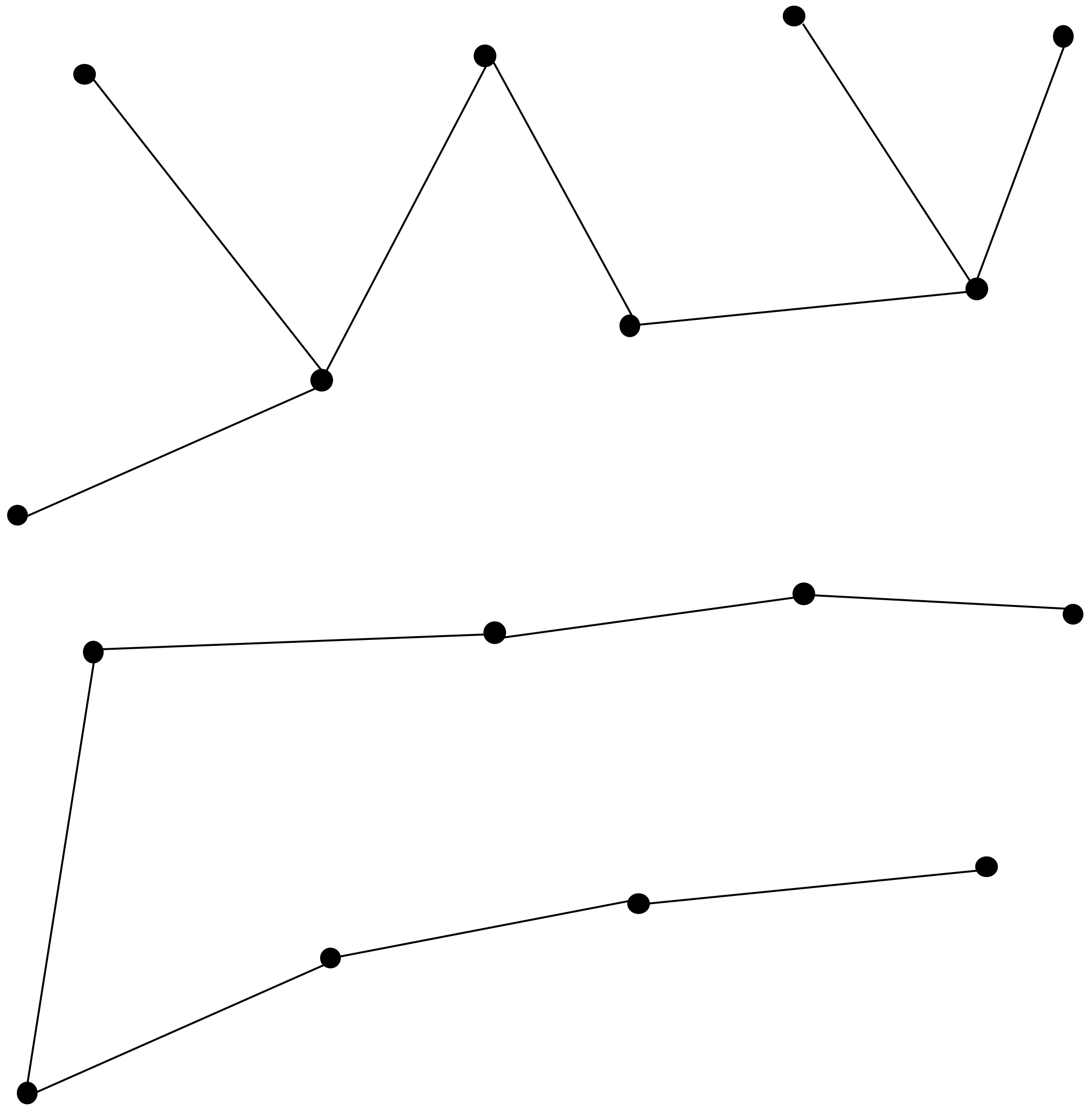
Redistricting as Graphs

- Iowa's current congressional district map doesn't split any counties! (only state where that is true)

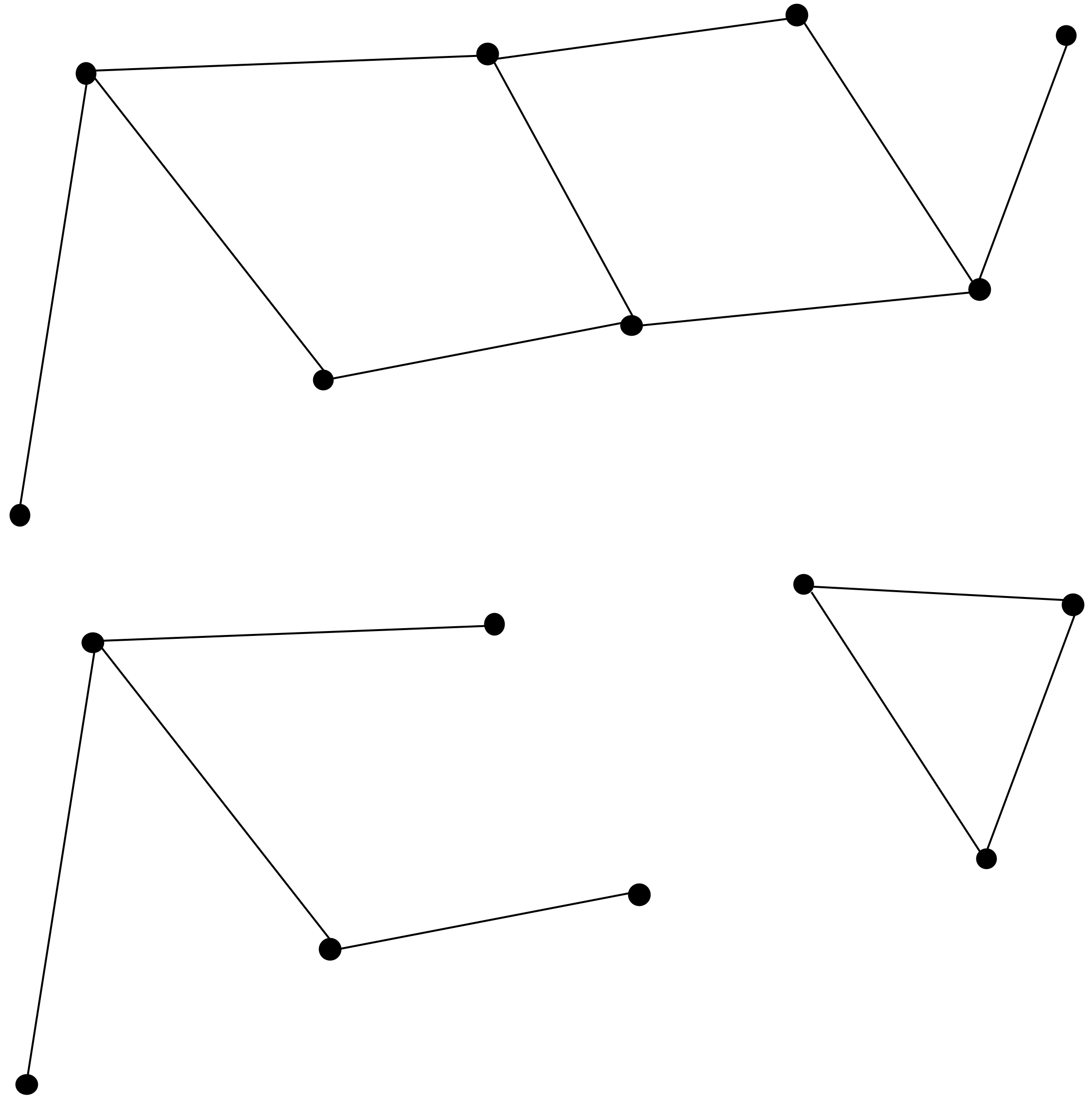




Spanning Trees



Not Spanning Trees



Spanning Trees

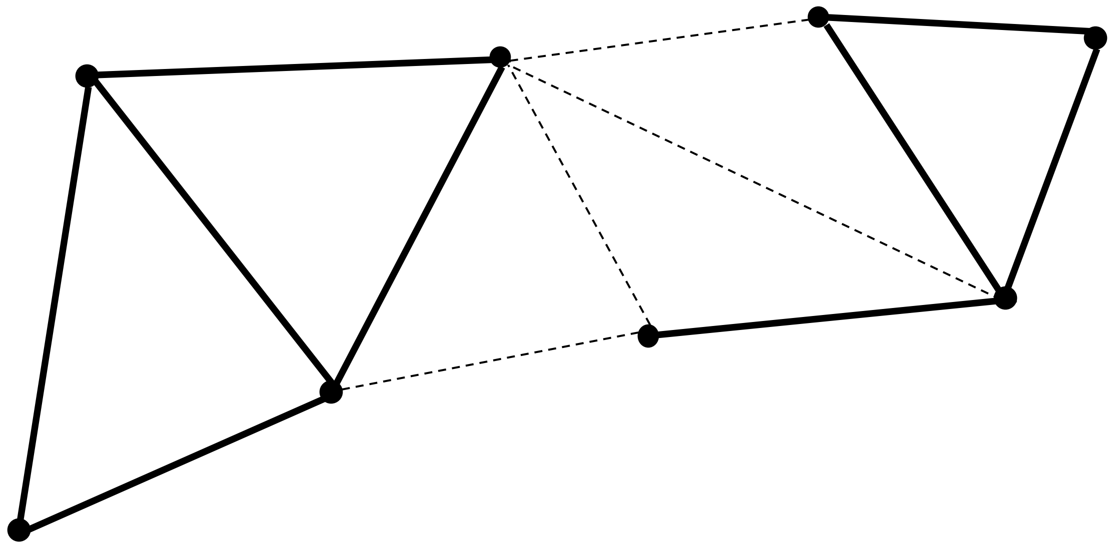
- Subgraph of a graph
- Connects all the vertices of the graph
- No loops
- Number of edges = (number of vertices) – 1

Generating New (random) Plans

- Flip Method – flip district assignment of random blocks
- Recombination Method – involves spanning trees

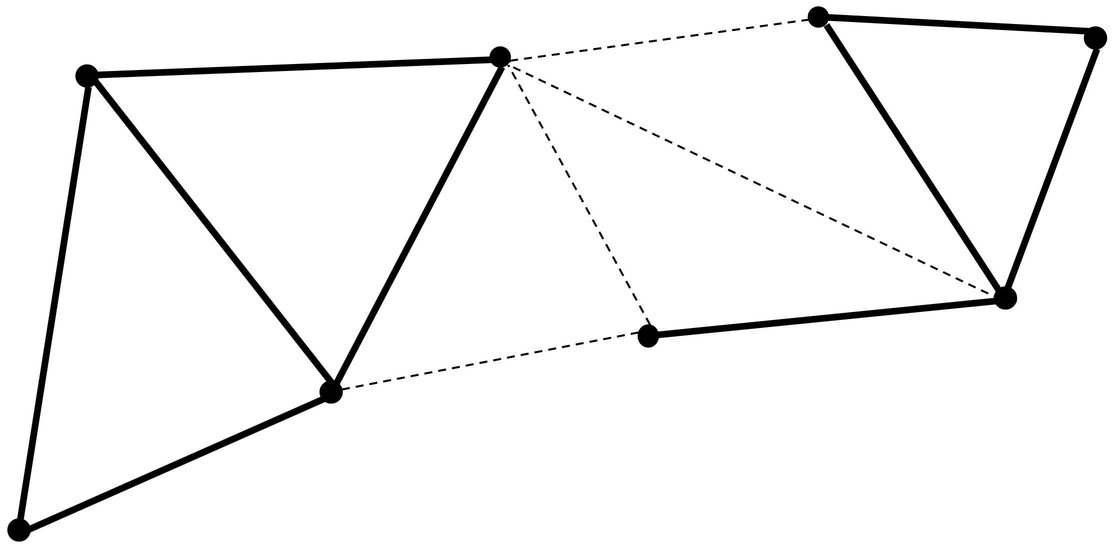
Recombination Method

- Suppose these are two neighboring districts and are part of a much larger grid.



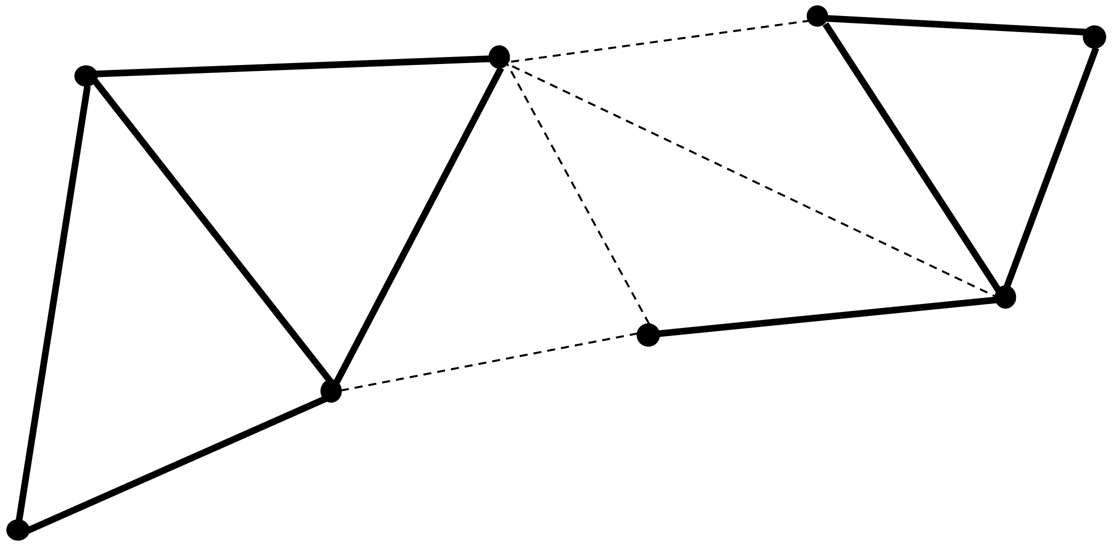
Recombination Method

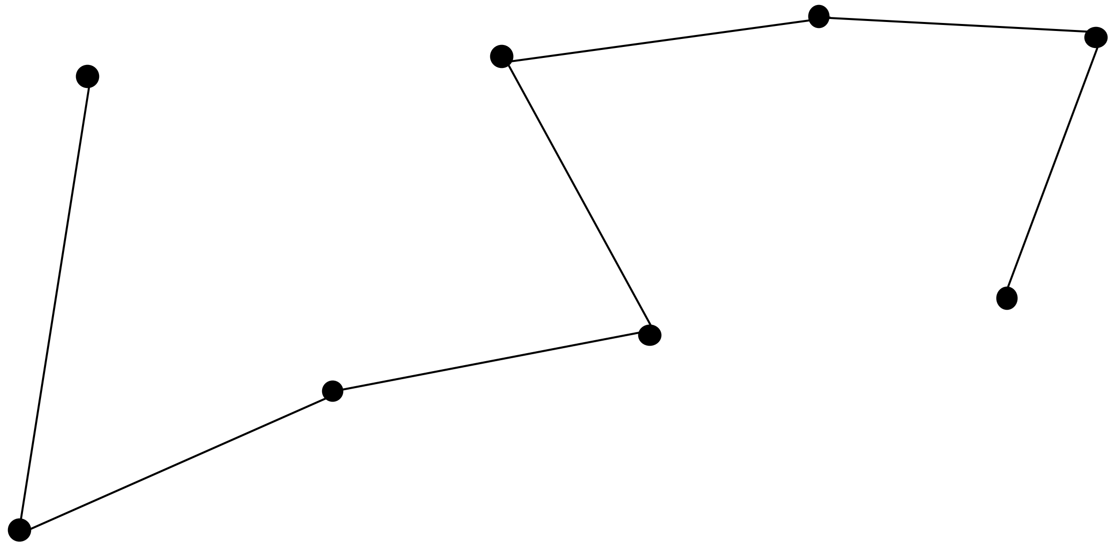
- Draw a random spanning tree that combines the two districts.



Recombination Method

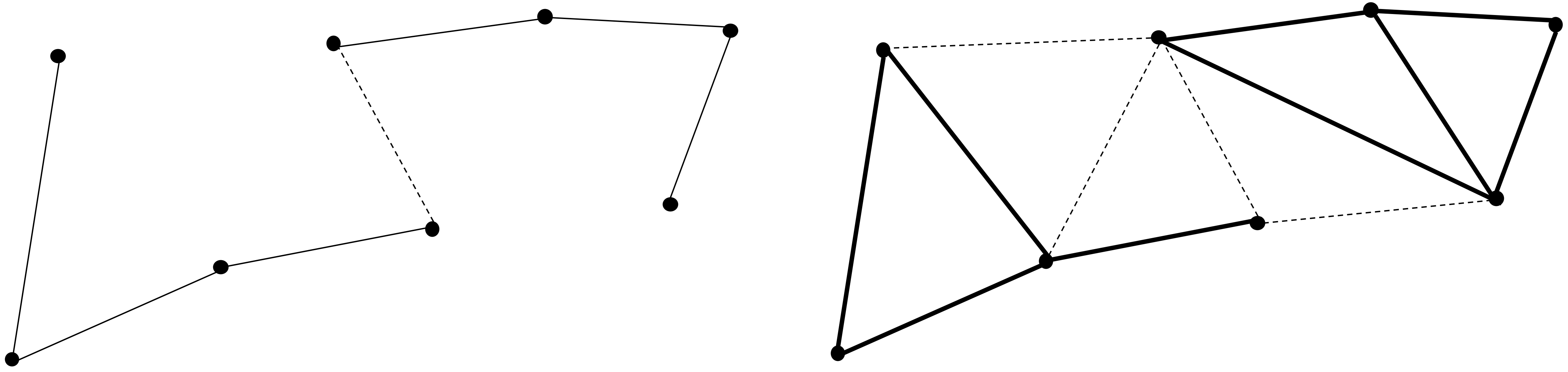
- Draw a random spanning tree that <u>combines</u> the two districts.





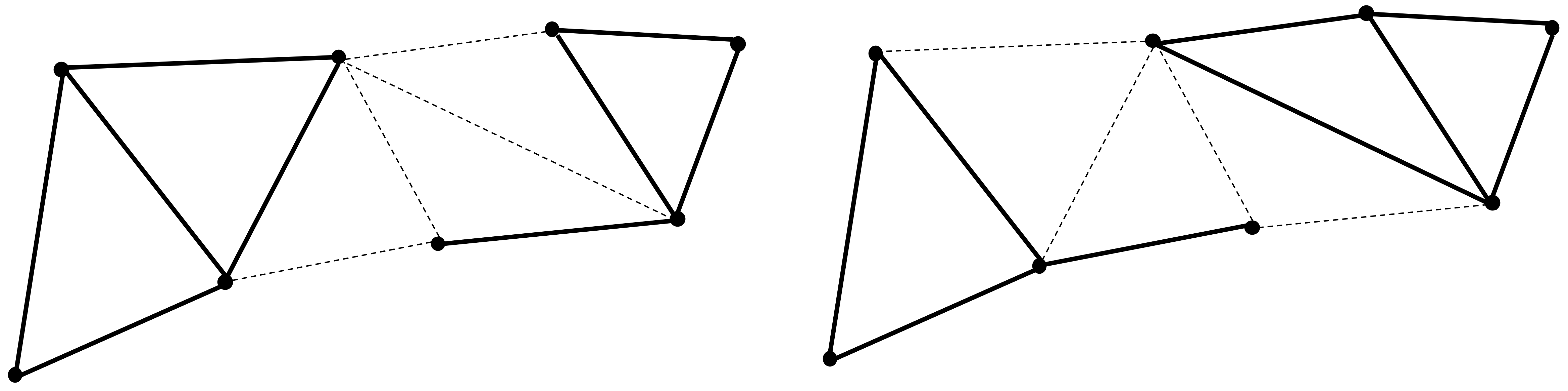
Recombination Method

- Remove the edge that splits the spanning tree into 2 separate spanning trees of equal length to form two new districts



Recombination Method

- Overall, two neighboring districts are randomly selected, combined, and separated into two new districts to form a new plan



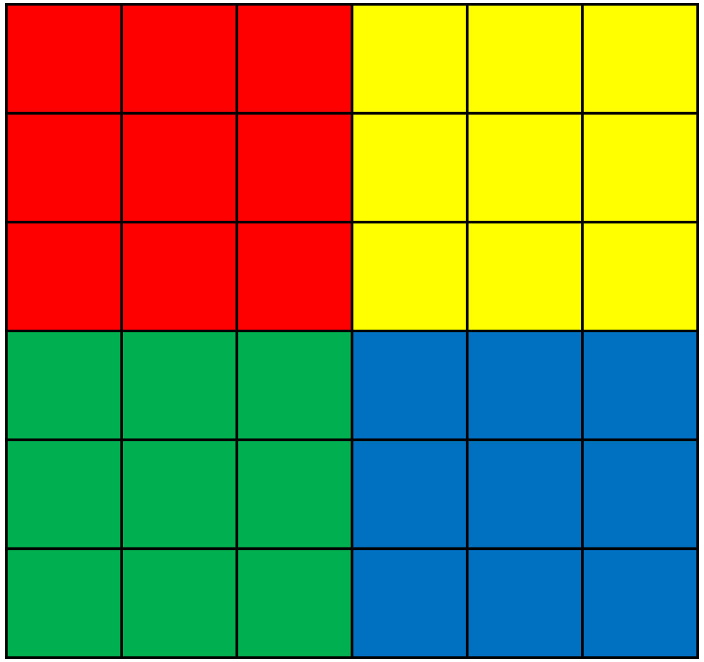
Recombination Method

- Start with actual map (look at party performance).
- Randomly select any two neighboring districts and randomly select a spanning tree that combines them.
- Remove the edge that separates them into two new equal population districts (not always possible, select a new spanning tree if this happens).
- Repeat this process thousands of times and build a distribution of party performance and see how the actual map compares.

- (Random walks) With enough trials, the distribution approaches the actual distribution no matter where you start!

Your Turn

- Apply 4 steps of the Recombination algorithm to the district grid below.



Flip vs. Recombination

Original
Arkansas
Congressional

After thousands
of Flips

After thousands
of ReComs

Flip vs. Recombination

- Flip Method seems to do much worse in terms of compactness (boundary points can't be switched!)
- Algorithms allow for modifications and restrict which maps will be allowed to be in the sample – setting constraints
- Add a compactness constraint
- Natural constraints of equal population and contiguity already built in
- In general, the computer algorithms perform a process (such as Flip or ReCom) and then checks it against criteria/constraints and accepts or rejects it as part of the sample and continues many times

Flip vs. Recombination

Algorithm adjusted to only accept plans with up to 5% of cut edges

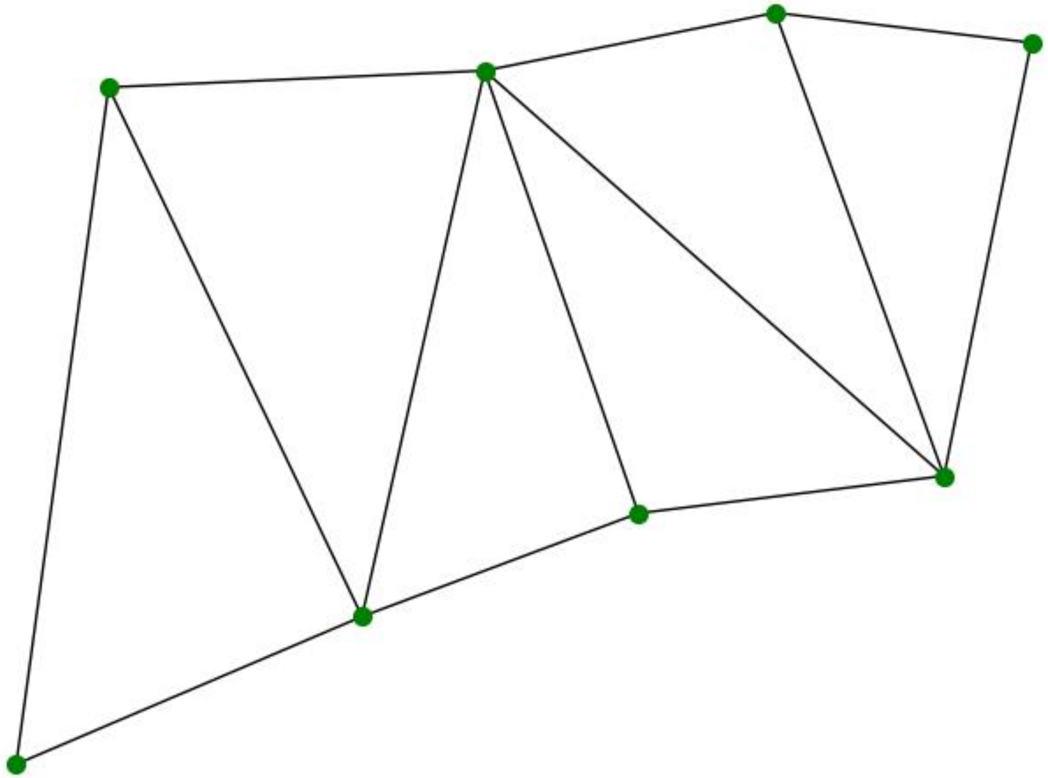
Original Arkansas Congressional

After 25,000 Flips

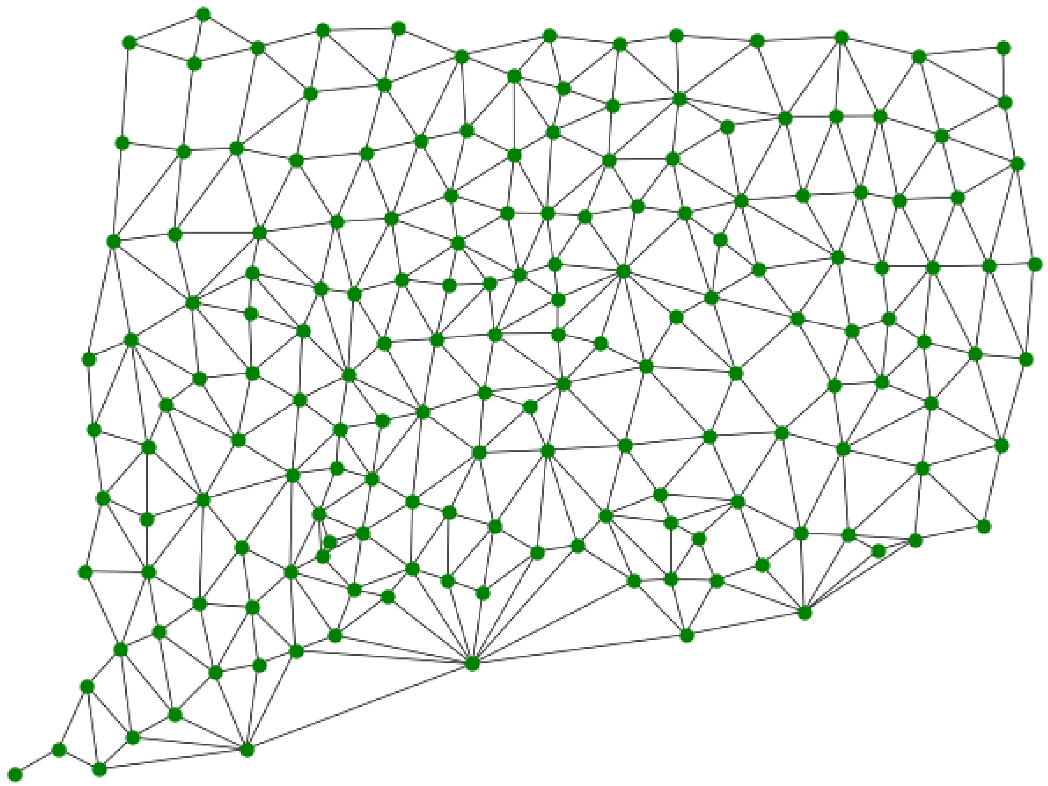
After 2,500 ReComs

Issue – Flip much slower to generate a good sample and more restrictions = harder to generate new/different plans

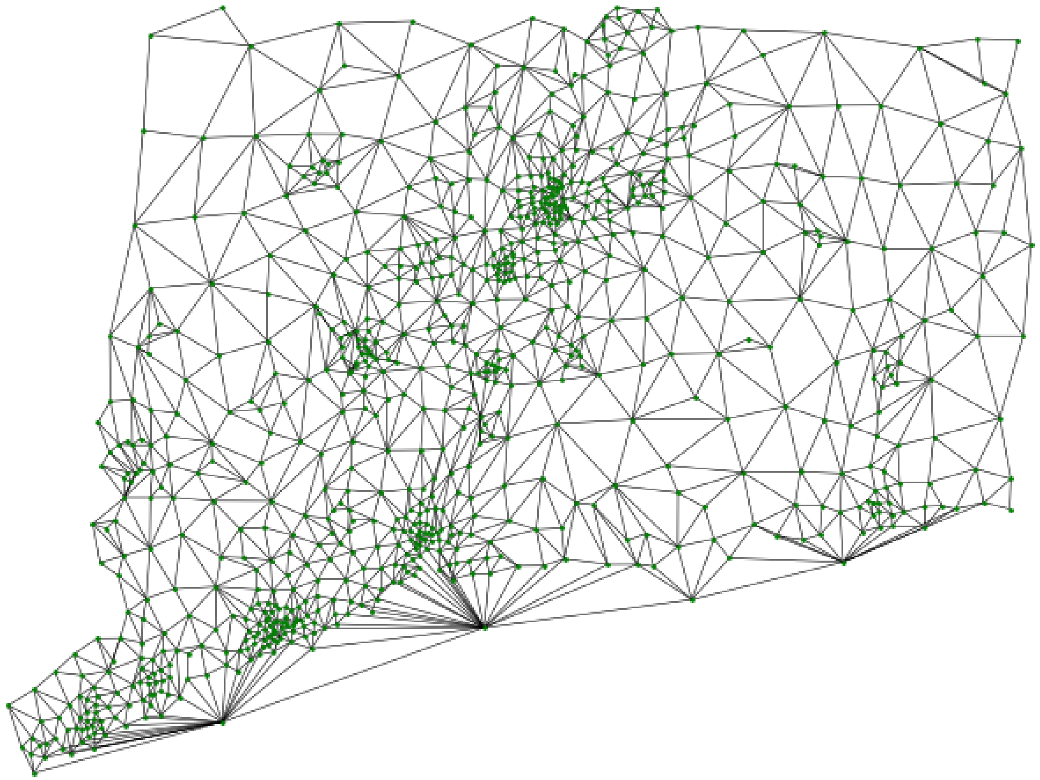
Graph of CT (County) – 8 points



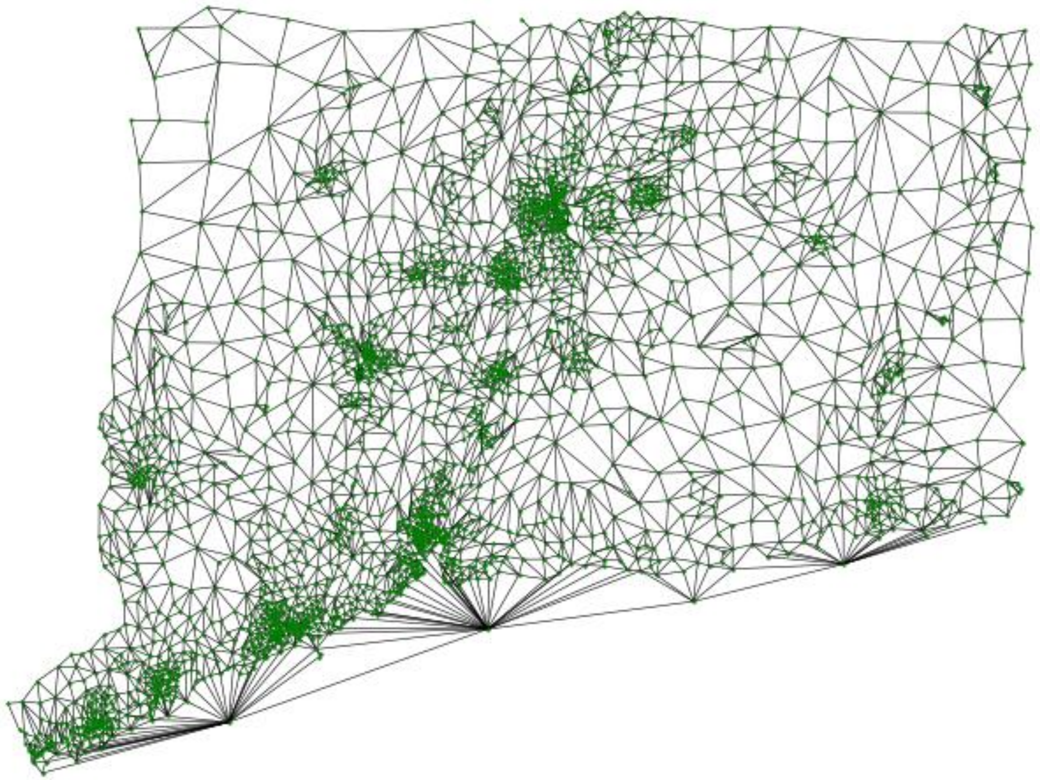
Graph of CT (County subunit) – 173 points



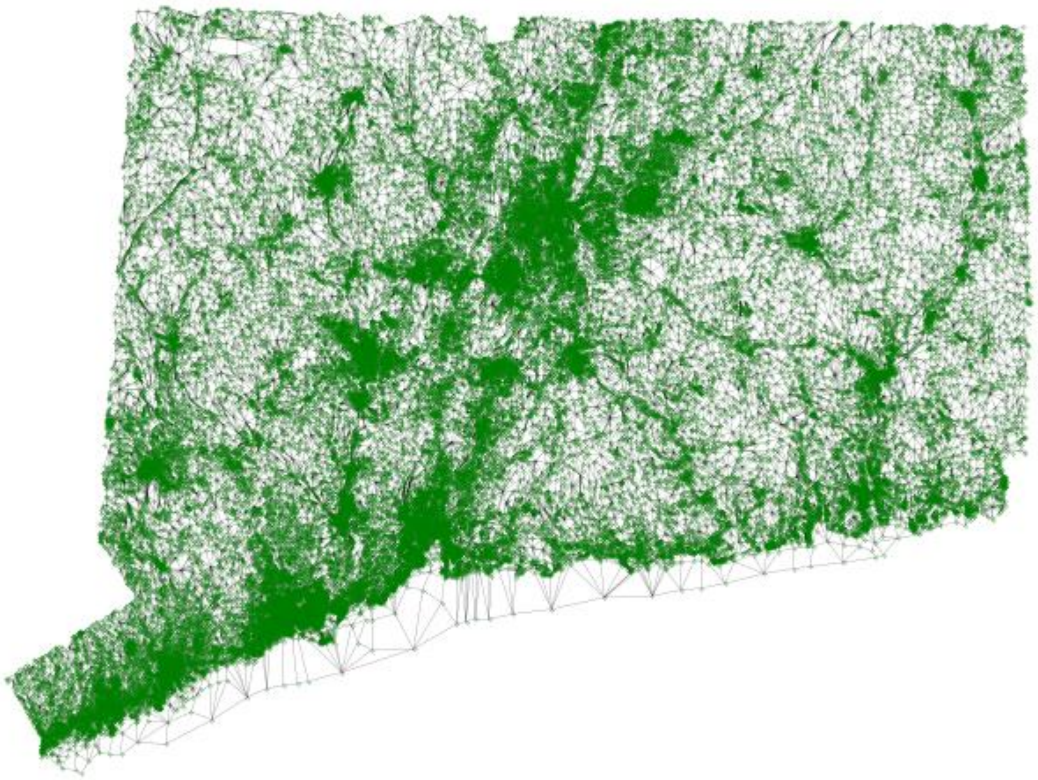
Graph of CT (Census tracts) – 833 points



Graph of CT (Block groups) – 2,585 points

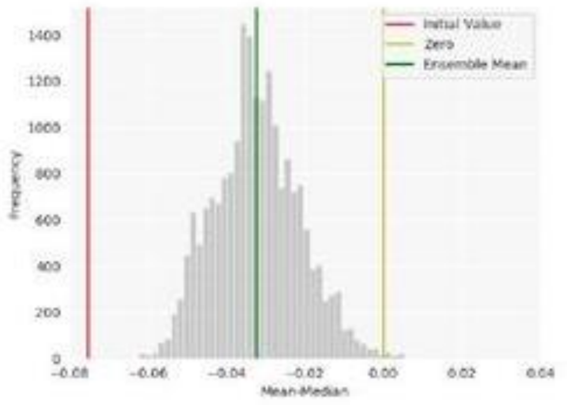


Graph of CT (Census blocks) – 67,578 points



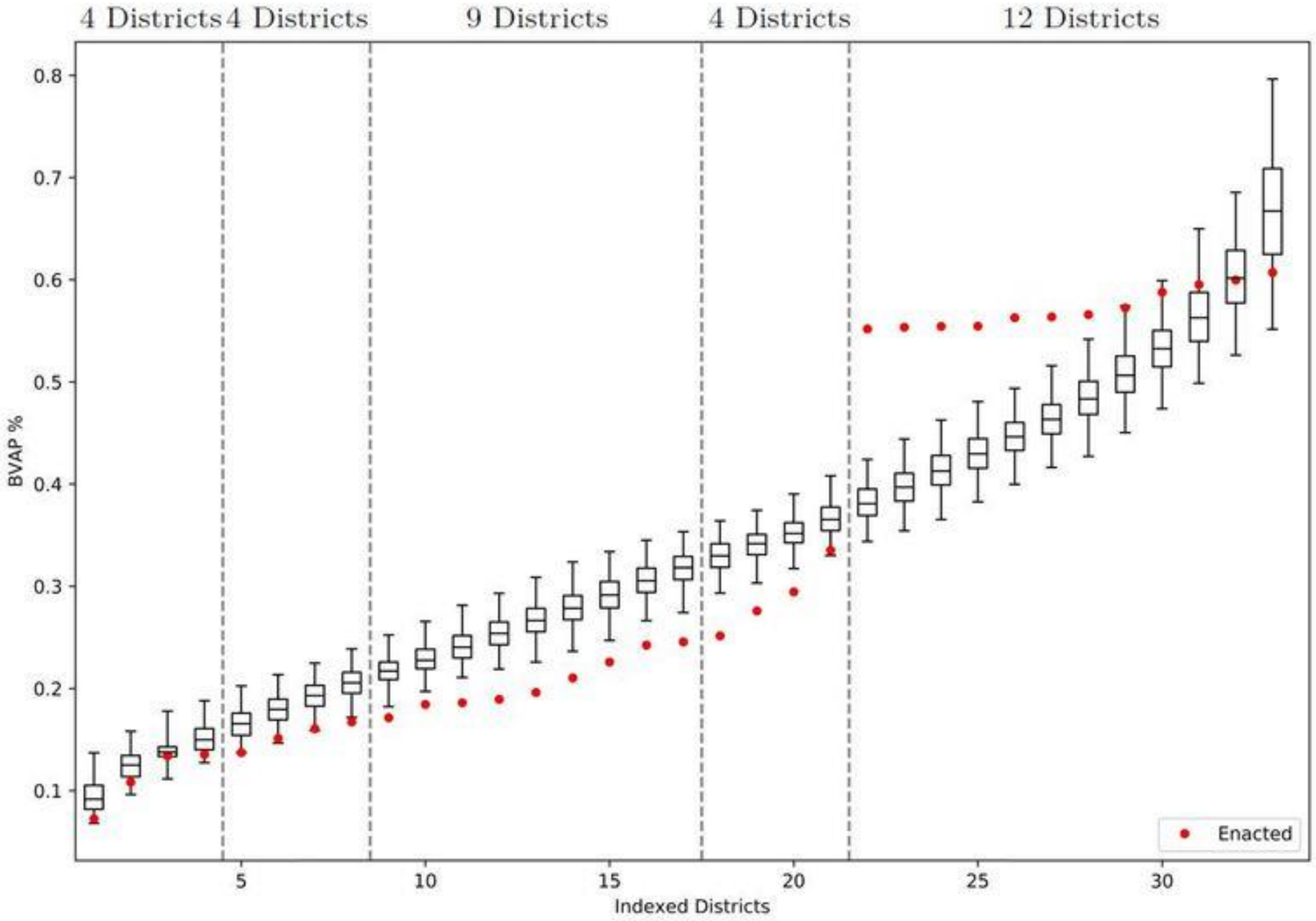
Interpretation

- Use a method/algorithm to build a large sample of possible district plans
- Calculate something of interest (such as efficiency gap) in all plans in the sample and then form a distribution and see where the actual plan lies
  - Is it an outlier? If yes, very strong evidence of gerrymandering
- Example at right: Mean-median difference of 10,000 plans in a sample formed by ReCom Method
- The actual plan (red) is the most extreme value in the sample, showing it would be so unlikely to occur by chance and it was done intentionally



(F) Enacted

Bethune-Hill v. Va. Board of Elections



MCMC Conclusion

- Impossible to consider all possible district plans so we use algorithms (Flip/ReCom) to create a representative sample of possibilities
- We look to see if a plan is an (extreme) outlier relative to the space of possibilities among various criteria
  - Partisan bias measures
  - Compactness
  - Demographics
- Creating a set of random plans is useful for <u>assessment</u>, but a random plan is not intended to be used for <u>enactment</u>
  - Can use human judgment to create maps and then assess them with these methods